I read the solution. It is a magical identity. The proof is long and mostly technical, and I hardly think that anyone will solve it during an exam with other questions.

The solution relies mostly on technical manipulations and Newton's Binomial Identity, but there are 2 interesting identites that are used during its proof, which can be given as cool problems themselves!

Before I will talk about those identites, note that there's a typo in the second equation, it should be:

$$\sum_{s=1}^{\infty} q^s 4^s \binom{s+2}{2} = (1-4q)^{-3} - 1$$

Identity 1:

$$\sum_{n\geq 0} \binom{n+2}{2} x^n = (1-x)^{-3}$$

It is a special case of the following identity:

$$\sum_{n\geq 0} \binom{n+\alpha-1}{\alpha-1} x^n = (1-x)^{-\alpha}$$

with $\alpha = 3$. There are two proofs for the general identity.

Proof 1: Let's start with Newton's binom: $\sum_{n\geq 0} \binom{\alpha}{n} x^n = (1+x)^{\alpha}$. This is true for $\alpha \in \mathbb{N}$. For other values of α this is also true - either as the definition (which satisfies the power rule $(1+x)^{\alpha}(1+x)^{\beta} = (1+x)^{\alpha+\beta}$ since $\sum_{i+j=n} \binom{\alpha}{i}\binom{\beta}{j} = \binom{\alpha+\beta}{n}$), or by noticing that the Taylor series of $(1+x)^{\alpha}$ is indeed $\sum_{n\geq 0} \frac{\alpha(\alpha-1)\cdots(\alpha-(n-1))}{n!} x^n = \sum_{n\geq 0} \binom{\alpha}{n} x^n$.

Now we can just plug $\alpha = -\beta$ and $y = -x$, and find that:

$$\sum_{n\geq 0} \binom{-\beta}{n} (-1)^n y^n = (1-y)^{-\beta}$$

And the following algebraic manipulation gives the result:

$$\binom{-\beta}{n} = \frac{(-\beta)(-(\beta+1))\cdots(-(\beta+n-1))}{n!} = (-1)^n \binom{\beta+n-1}{n} \implies$$

$$\begin{aligned} \sum_{n\geq 0} \binom{\beta+n-1}{\beta-1} y^n &= \sum_{n\geq 0} \binom{\beta+n-1}{n} y^n \\ &= \sum_{n\geq 0} \binom{-\beta}{n} (-y)^n \\ &= (1-y)^{-\beta} \end{aligned}$$

Proof 2: The second proof is combinatorial:

$$\begin{aligned}(1-x)^{-\alpha} &= \underbrace{\frac{1}{1-x}\cdots\frac{1}{1-x}}_{\alpha \text{ times}} \\ &= \underbrace{\sum x^i \cdots \sum x^i}_{\alpha \text{ times}}\end{aligned}$$

So the coefficient of x^n in $(1-x)^{-\alpha}$ is the number of solutions to $a_1 + a_2 + \cdots + a_\alpha = n$ with $a_i \geq 0$, which is $\binom{\alpha+n-1}{\alpha-1}$.

Identity 2: Let C_n be the n'th Catalan number: $\frac{1}{n+1}\binom{2n}{n}$, and let $C(x)$ be its generating function $\sum C_i x^i$. It is well-known that $C(x) = 1 + xC^2(x)$, which gives $C(x) = \frac{1-\sqrt{1-4x}}{2x}$. The following marvelous identity is true for any non-negative integer t:

$$\sum_{n\geq 0}\binom{2n+t}{n}x^n = \frac{C^t(x)}{\sqrt{1-4x}}$$

Proof 1 (Inductive): Call the LHS by the name L_t and the RHS by the name R_t.

Note that both L_t and R_t satisfy the same recursion of degree 2:

$$C(x) = 1 + xC^2(x) \implies C^{n+1}(x) = C^n(x) + xC^{n+2}(x) \implies R^{n+1} = R^n + xR^{n+2}$$

We also have:

$$\begin{aligned}L_{t+1} &= \sum(\binom{2n+t}{n} + \binom{2n+t}{n-1})x^n \\ &= \sum\binom{2n+t}{n}x^n + \sum\binom{2(n-1)+t+2}{n-1}x^n \\ &= L_t + x\sum\binom{2(n-1)+t+2}{n-1}x^{n-1} \\ &= L_t + xL_{t+2}\end{aligned}$$

So it is enough to prove the identity for $t = 0, 1$. For $t = 0$ it follows from Newton's binom applied to $(1-4x)^{-1/2}$, since $\binom{2n}{n} = (-4)^n\binom{-1/2}{n}$.

For $t = 1$ it can be proved by algebraic manipulations:

$$\begin{aligned}\sum_{n\geq 0}\binom{2n+1}{n}x^n &= \sum_{n\geq 0}C_n(2n+1)x^n \\ &= C(x) + 2xC'(x)\end{aligned}$$

Subtituing $C(x) = \frac{1-\sqrt{1-4x}}{2x}$, this expression simplifies to $\frac{1-\sqrt{1-4x}}{2x\sqrt{1-4x}}$ which is $\frac{C(x)}{\sqrt{1-4x}}$ (note that $C'(x) = \frac{1}{x\sqrt{1-4x}} - \frac{1-\sqrt{1-4x}}{2x^2}$).

Proof 2 (Combinatorial): As we've seen, for $t = 0$ this is a consequence of Newton's binom. Suppose we want to prove this by induction on t. To pass from t to $t+1$, we need to show:

$$C(x)\sum \binom{2n+t}{n}x^n \quad = \quad \sum \binom{2n+t+1}{n}x^n$$

Comapring coefficients, it becomes:

$$\sum_{i+j=n} C_i \binom{2j+t}{j} \quad = \quad \binom{2n+t+1}{n}$$

Or:

$$\sum_{i+j=n} \frac{\binom{2i}{i}\binom{2j+t}{j}}{i+1} \quad = \quad \binom{2n+t+1}{n}$$

This is an extremely cool identity in my opinion. It is a special case of the following more general identity from the 18th century, proved by Rothe (it is called "Rothe's Convolution"):

$$(A) \sum_{i+j=n} \frac{a}{a+ib}\binom{a+ib}{i}\frac{c}{c+jb}\binom{c+jb}{j} \quad = \quad \frac{a+c}{a+c+bn}\binom{a+c+bn}{n}$$

$$(B) \sum_{i+j=n} \frac{a}{a+ib}\binom{a+ib}{i}\binom{c+jb}{j} \quad = \quad \binom{a+c+bn}{n}$$

If you plug $b = 0$, you get a famous identity: $\sum \binom{a}{i}\binom{c}{j} = \binom{a+c}{n}$. If you plug $a = 1, b = 2, c = t$ in identity (B), we get our desired identity.

There's a geometric proof for those 2 identites:
Let $A_n(a,b) = \frac{a}{a+bn}\binom{a+bn}{n}$.

1. The number of lattice paths from $(0,0)$ to $(c+nd, n)$ not touching the line $(c+yd, y)$ except at $y = n$ is $A_n(c, d+1)$. (This is the hard part, but we only need it for $d = 1$, which is easier but still hard).

2. The first identity can be written as $\sum A_i(a,b)A_j(c,b) = A_n(a+c,b)$. By splitting the paths counted by $A_n(a+c,b)$ into 2 parts (part before it touches the line $(a+(b-1)y, y)$, and part after), we get identity (A).

3. If we consider all the lattice paths from $(0,0)$ to $(a+c+n(b-1), n)$, we get $\binom{a+c+bn}{n}$. If we split those paths into 2 parts as we did in identity (A), we get identity (B).